Southwest Fisheries Center Administrative Report No. 17H, 1978

IMPROVED BOUNDS FOR ROW AGGREGATED LINEAR PROGRAMS:

A NOTE

Roy Mendelssohn
Southwest Fisheries Center
National Marine Fisheries Service, NOAA
Honolulu, HI 96812

July 1978

DRAFT FOR COMMENT

For any partition $\rho' = \{R'_\ell : \ell = 1, \ldots, L'\}$, let $\{e_1 \ldots, e_m\}$ be known positive numbers, and $\{q_1, \ldots, q_{L'}\}$ known nonnegative numbers, such that u^*, the optimal dual variables in (1) satisfy:

$$\sum_{i \in R'_\ell} e_i u_i^* \leq q_\ell \quad , \quad \ell = 1, \ldots, L' \quad . \tag{3}$$

Then Zipkin [3] shows:

$$\hat{z} - \xi_\alpha^- \leq z^*$$

where $$\xi_\alpha^- = \sum_{\ell=1}^{L'} \left[\max_{i \in R'_\ell} \{(A_i \hat{x} - b_i)/e_i\}\right]^+ q_\ell$$

where $\hat{x}$ is an optimal solution to (2).

To extend theorem 2 in [1] and theorem 2.1 in [2], consider the dual to (1):

$$\begin{aligned} z^* = \text{minimize } & ub \\ \text{s.t. } & uA \geq c \\ & u \geq 0 \end{aligned} \tag{4}$$

transformed to:

$$\begin{aligned} \text{maximize } & u(-b) \\ \text{s.t. } & u(-A) \leq -c \\ & u \geq 0 \end{aligned} \tag{5}$$

The partition ρ and ρ', and the bounds (3) now apply to a column aggregated LP. The equivalent aggregate problem is:

$$\begin{aligned} &\text{maximize } U(-\tilde{b}) \\ &\quad \text{s.t. } U(-\hat{A}) \le -c \\ &\qquad\qquad U \ge 0 \end{aligned} \tag{6}$$

where U is an L-vector of variables. From proposition 2 in [3], and theorem 2.1 in [2], there exists a function $z(\theta)$, such that at θ^*,

$$\bar{u}(-b) \le z(\theta^*) \le \hat{U}(-\tilde{b}) + \xi_\alpha^- \tag{7}$$

This is equivalent to:

$$\hat{z} - \xi_\alpha^- \le -z(\theta^*) \le \bar{u}b = z^* \tag{8}$$

Equation (8) states that Kallio's method, as used to tighten the upper bounds on column aggregate problems, can also be used to tighten the lower bounds for a row aggregated problem. The reader is referred to [1] and theorem 2.1 in [2] for details of how to calculate $-z(\theta^*)$.

LITERATURE CITED

[1] M. Kallio. "Computing bounds for the optimal value in linear programming." Naval Research and Logistic Quarterly 24(2): 301-308. (1977)

[2] R. Mendelssohn. "Improved bounds for aggregated linear programs." SWFC Admin. Rep. 16H, 1978, 19 p. (1978)

[3] P. Zipkin. "Aggregation in linear programming." Ph.D. Dissertation, Yale University, New Haven, 189 p. (1977)

In [2], a result of Kallio [1] is extended in order to tighten Zipkin's [3] upper bound for column aggregated linear programs (LPs). In this note, it is shown that an identical procedure can be used to tighten Zipkin's lower bound for row aggregated LP's. The notation follows [3] throughout, and familiarity with the results and methods of [1, 2, 3] is assumed.

The primal LP to be solved is:

$$\begin{aligned} z^* = \text{maximize} \quad & cx \\ \text{subject to } & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{1}$$

where $c = (c_j)$ is an n-vector, $b = (b_i)$ is an m-vector, $A = (a_{ij})$ is an $m \times n$ matrix, and $x = (x_j)$ is an n-vector of variables.

Let $\rho = \{R_\ell : \ell = 1, \ldots, L\}$ be a partition of $\{1, \ldots, m\}$, where $|R_\ell| = m_\ell$. Let f be a nonnegative m -vector whose components sum to unity. Define:

$$\tilde{b} = (\tilde{b}_\ell); \ \tilde{b}_\ell = f^\ell b^\ell$$

$$\hat{A} = (\hat{a}_{kj}); \ \hat{a}_{kl} = f^\ell A_\ell^k$$

so that the row aggregated LP becomes:

$$\begin{aligned} z = \max\ & cx \\ \text{subject to } & \hat{A}x \leq \tilde{b} \\ & x \geq 0 \end{aligned} \tag{2}$$

Define $K(\theta, k)$ as:

$$K(\theta, k) = \begin{cases} \theta_k^+ & \text{if } G^k(x, y) - \theta h_k^+ \geq G^k(x, y) - \theta h_k^- \\ \theta_k^- & \text{otherwise} \end{cases}$$

and define the set $K(\theta)$ as

$$K(\theta) = \Big\{ k \,\big|\, K(\theta, k) > \theta \text{ and } K(\theta, k) = \theta_k^- \text{ or}$$

$$K(\theta, k) \leq \theta \text{ and } K(\theta, k) = \theta_k^+ \Big\}$$

Let the function $v(\theta)$ be defined as:

$$v(\theta) = \bar{z}\theta + \sum_{k \in K(\theta)} \left(G^k(x, y) - \theta\, w_k \right) p_k$$

where $$w_k = \begin{cases} h_k^+ & \text{if } K(\theta, k) = \theta_k^+ \\ h_k^- & \text{otherwise} \end{cases}$$

and minimize $v(\theta)$ with respect to θ using the following marginal analysis:

1) Choose: $\theta \in \left\{ \theta_1^+ \; \theta_1^-, \ldots, \theta_k^+, \theta_k^- \right\}$

2) Evaluate: $v_-(\theta) = \bar{z} - \sum_{k \in K(\theta)} w_k p_k$

3) If $v_-(\theta)$ equals zero or changes sign at θ, then set $\theta^+ = \theta$ and go to 4). Else, if $v_-(\theta)$ is negative, increase θ to the value of the next largest element of $\{\theta_1^+, \theta_1^-, \ldots, \theta_k^+, \theta_k^-\}$. If $v_-(\theta)$ is positive, decrease

U.S. DEPARTMENT OF COMMERCE
National Oceanic and Atmospheric Administration
NATIONAL MARINE FISHERIES SERVICE
Southwest Fisheries Center
Honolulu Laboratory
P. O. Box 3830
Honolulu, Hawaii 96812

July 14, 1978

Enclosed is a short companion to the earlier paper I sent you, "Improved bounds for aggregated linear programs." The purpose of the note was to convince myself that the method to improve upper bounds for column aggregated LP's does work for row aggregated LP's. Again, comments, suggestions, etc., are most welcome.

With regards,

Roy

Roy Mendelssohn
Operations Research Analyst

Identical letters to:

Professor P. Zipkin
Graduate School of Business
Uris Hall 416
Columbia University
New York, N.Y. 10027

Fwd

Professor M. J. Sobel
c/o Bell Laboratories
Bldg. WB, Room 1G-305
Crawford Corners Road
Holmdel, N.J. 07733

Professor H. Gold
Director
Biomathematics Program
Department of Statistics
North Carolina State University
Box 5457
Raleigh, North Carolina 27650

Professor E. L. Porteus
Graduate School of Business
Stanford University
Stanford, CA 94305